A correction to Epp's paper "Elimination of wild ramification"

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Abstract. We fill a gap in the proof of one of the central theorems in Epp's paper, concerning p-cyclic extensions of complete discrete valuation rings.

In his famous paper [1], Epp considers the following situation: S and R are two discrete valuation rings such that (1) S dominates R, and (2) if the characteristic p of the residue field of S is not zero, then its largest perfect subfield is separable and algebraic over the residue field of R. He proves that then, there exists a discrete valuation ring T which is a finite extension of R such that the localizations of the normalized join of S and T are weakly unramified over T. Towards this result, he proves the following theorem, assuming that all discrete valuation rings are complete:

Theorem (1.3). *Let S be a p-cyclic extension of S_0 where S_0 is a weakly unramified extension of R such that $\mathfrak{L}^{p^\infty} = \mathfrak{K}$, where $\mathfrak{L}$ and $\mathfrak{K}$ are the residue fields of S_0 and R respectively. There exists a finite extension T of R such that TS is weakly unramified over T.*

There is a mistake in the proof of the *Equal characteristic $p \neq 0$ case*. We will sketch those parts of the proof that are necessary to understand and correct the mistake.

Using well-known structure theorems that are discussed in section 0.1 of his paper, Epp writes $R = \mathfrak{K}[[\pi]]$ and $S_0 = \mathfrak{L}[[\pi]]$, where π is a local parameter of R. By Artin-Schreier theory and the fact that the Artin-Schreier polynomial $X^p - X$ is additive and surjective on the maximal ideal of the power series ring S_0 , Epp finds that the p-cyclic extension S of S_0 is defined by an equation of the form

$$z^p - z \;=\; a_{-N}\pi^{-N} + \ldots + a_{-1}\pi^{-1} + a_0 \tag{1}$$

with $a_i \in \mathfrak{K}$. In the case of $N = 0$ there is nothing to prove, so we assume that $N \neq 0$. Epp defines the following subsets of $\{1, \ldots, N\}$:

$$I = \{m \mid a_{-m} \in \mathfrak{K}, a_{-m} \neq 0\} \quad \text{and} \quad J = \{m \mid a_{-m} \notin \mathfrak{K}, a_{-m} \neq 0\} .$$

After dealing with the case of $J = \emptyset$, Epp assumes that $J \neq \emptyset$. The idea is now to find some $d \in \mathfrak{K}[[t]]$, where $t^{p^k} = \pi$ for some k, such that after replacing z by $z - d$ and adding $d - d^p$ to the right hand side of equation (1), the defining equation (of $\mathfrak{K}[[t]]S$ over $\mathfrak{K}[[t]]$) will be of the form

$$z^p - z = ct^{-n} + \ldots$$

where n is divisible by p, but $c \notin \mathfrak{L}^p$. Epp shows that then, $\mathfrak{K}[[t]]S$ is weakly unramified over $\mathfrak{K}[[t]]$, so we can take $T = \mathfrak{K}[[t]]$.

Note that a transformation of the above type replaces a p-th power d^p on the right hand side of (1) by its p-th root d. By a repeated application of such replacements, Epp seeks to get rid of all coefficients that lie in $\mathfrak{L}^p \setminus \mathfrak{K}$. To this end, he chooses a positive integer ν such that

$$\min_{m \in J} mp^\nu \; > \; \max_{m \in I} m .$$

Since $\mathfrak{L}^{p^\infty} = \mathfrak{K}$,

$$\nu_m \; := \; \max\{i \mid a_{-m} \in \mathfrak{L}^{p^i}\} \; < \; \infty$$

for every $m \in J$. Let $\mu := \max\{\nu_m \mid m \in J\}$, and let t be such that $t^{p^{\nu+\mu}} = \pi$. Equation (1) can now be written

$$z^p - z \; = \; \sum_{m \in J} a_{-m} t^{-mp^{\nu+\mu}} \; + \; \sum_{s \in I} a_{-s} t^{-sp^{\nu+\mu}} \; + \; a_0 \; .$$

Using the above described transformations, Epp arrives at a defining equation

$$z^p - z \; = \; \sum_{m \in J} c_{-m} t^{-mp^{\nu+\mu-\nu_m}} \; + \; \sum_{s \in I} c_{-s} t^{-s} \; + \; a_0 \; , \qquad (2)$$

where:
- for every $s \in I$, $c_{-s} \in \mathfrak{K}$ is the $p^{\nu+\mu}$-th root of $a_{-s} \in \mathfrak{K}$ (note that $\mathfrak{K}$ is perfect!),
- for every $m \in J$, $c_{-m} \in \mathfrak{L} \setminus \mathfrak{L}^p$ is the p^{ν_m}-th root of a_{-m} (recall that $a_{-m} \in \mathfrak{L}^{p^{\nu_m}} \setminus \mathfrak{L}^{p^{\nu_m+1}}$),
- for every $m \in J$, p divides $mp^{\nu+\mu-\nu_m}$ since $\nu \geq 1$ and $\mu \geq \nu_m$,
- for every $m \in J$ and $s \in I$, $-s > -mp^{\nu+\mu-\nu_m}$ since $s < mp^\nu$ by the choice of ν.

Now Epp claims that the term in t with the most negative exponent has a coefficient which is not in $\mathfrak{L}^p$. This is not necessarily true.

It would hold if the exponents $-mp^{\nu+\mu-\nu_m}$ were distinct, for distinct m. But this could be false since we know nothing about the ν_m .

Example. Suppose that

$$z^p - z \;=\; a_{-m_0}\pi^{-m_0} + a_{-m_1}\pi^{-m_1} + \ldots$$

where $m_0 = m_1 p$, $a_{-m_0} = c_0^p$ and $a_{-m_1} = c_1 - c_0$ with $c_0 \in \mathfrak{L} \setminus \mathfrak{L}^p$ and $c_1 \in \mathfrak{L}^p$. Then $a_{-m_0} \notin \mathfrak{K}$, $a_{-m_1} \notin \mathfrak{L}^p$, $\nu_{m_1} = 0$, $\nu_{m_0} = 1$, and $a_{-m_1} + a_{-m_0}^{1/p} = a_{-m_1} + c_0 = c_1 \in \mathfrak{L}^p$. Using the notation of (2), we find that $-m_0 p^{\nu+\mu-\nu_{m_0}} = -m_1 p^{\nu+\mu-\nu_{m_1}}$ and $c_{-m_0} + c_{-m_1} = c_1 \in \mathfrak{L}^p$. □

So we see that the coefficient of the term in t with the most negative exponent can well lie in $\mathfrak{L}^p$. Choosing c_1 to lie in $\mathfrak{K}$ in our example, we see that this coefficient may even lie in $\mathfrak{K}$, so that the corresponding exponent "switches" from the set J to the set I. However, whenever such a recombination happens and we start over with the new equation (2), the new set J will be smaller than the original set J. So the gap in Epp's proof can be closed by repeating his transformations until his assertion is satisfied or the set J is empty.

The latter may well happen: consider the equation

$$z^p - z \;=\; a_{-m_0}\pi^{-m_0} + a_{-m_1}\pi^{-m_1} + a_0$$

with the conditions of our example, and assume in addition that $c_1 = 0$. Then the transformation leads to the equation

$$z^p - z \;=\; a_0\ .$$

This shows that even if $J \neq \emptyset$ in the original equation (1), the residue field extension of TS over T may end up to be an Artin-Schreier extension, in contrast to the purely inseparable extension which Epp obtains for this case.

A far-reaching generalization of Epp's results will be proved in [2].

References

1. Epp, Helmut P.: Eliminating Wild Ramification. Invent. Math. **19** (1973) 235–249
2. Kuhlmann, F.–V.: The generalized stability theorem and henselian rationality of valued function fields. In preparation